

On Urysohn Biclosure Spaces



Halgwrđ M. Darwesh*, Alias B. Khalaf**

* Faculty of Science and Science Educations/School of Science /Mathematics department, University of Sulaimani, Sulaimani, Kurdistan Region, Iraq, e-mail:

darweshymath@yahoo.com

** Faculty of Science / Mathematics department, Duhok University, Duhok, Kurdistan Region, Iraq, e-mail: aliasbkhalaf@gmail.com

Abstract:

The purpose of this paper is to introduce the concept of Urysohn biclosure space; also we investigate some of its basic properties and relationships with some other known space such as T_2 - biclosure space, regular biclosure space.

Keywords: closure operator; closure space; biclosure space; urysohn biclosure space.

Introduction:

Closure spaces were introduced by Cech [1]. The notion of closure operators are very useful tool in several areas of classical mathematics. They play an important role in general topology, Boolean algebra and digital topology ([1], [2] and [3]). Kelly in [4], introduced the notion of bitopological spaces such spaces are equipped with two arbitrary topologies. He extended some of the standard results of separation axioms in a topological space to a bitopological space. So many papers have been written to generalize topological concepts to bitopological setting. Boonpok [5] introduced the notion of biclosure spaces. Such spaces equipped with two arbitrary closure operators. He extended the concepts of hausdorffness [6], and regularity [7] to biclosure setting. In this paper, we introduce and study the concept of urysohn biclosure spaces and give some of their properties.

Preliminaries:

In this section we recall some definitions and results, which are used in the sequel. First we recall the following definitions and results from [2].

Definition 2.1 A map $c: P(X) \rightarrow P(X)$ defined on a power set $P(X)$ of a set X is called a closure operator on X and the pair (X, c) is called a closure space if the

following axioms are satisfied:

C1. $c\phi = \phi$.

C2. $A \subseteq cA$ for every $A \subseteq X$.

C3. for all $A, B \subseteq X$, if $A \subseteq B$, then $cA \subseteq cB$.

A closure operator c on a set X is called additive (resp. idempotent) if $A, B \subseteq X$, then $c(A \cup B) = cA \cup cB$ (resp. $cA = ccA$).

Definition 2.2 A subset A of a closure space (X, c) is called closed if $cA = A$, it is open if its complement in X is closed, and its clopen if its both open and closed in X .

Definition 2.3 Let (X, c) and (Y, e) be two closure spaces. A function $f: X \rightarrow Y$ is said to be continuous if $f(cA) \subseteq ef(A)$, for every subset A of X .

Proposition 2.4 Let (X, c) and (Y, e) be two closure spaces. A function $f: X \rightarrow Y$ is continuous if and only if $cf^{-1}(B) \subseteq f^{-1}(eB)$ for every subset B of Y . Furthermore, if $f: X \rightarrow Y$ is continuous, then $f^{-1}(F)$ is closed in (X, c) , for each closed subset F of (Y, e) .

Definition 2.5 The product of a family of $\{(X_\lambda, c_\lambda): \lambda \in \Lambda\}$ of closure spaces, denoted by $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda)$, is the closure space $(\prod_{\lambda \in \Lambda} X_\lambda, c)$, where $\prod_{\lambda \in \Lambda} X_\lambda$ denotes the cartesian product of sets X_λ , $\lambda \in \Lambda$, and c is the closure operator generated by the projections $\pi_\lambda: \prod_{\lambda \in \Lambda} X_\lambda \rightarrow X_\lambda, \lambda \in \Lambda$, i.e., is defined by $cA = \prod_{\lambda \in \Lambda} c_\lambda \pi_\lambda(A)$ for each

$$A \subseteq \prod_{\lambda \in \Lambda} X_\lambda.$$

The following definitions and results are from [5], [6], [7] and [9].

Definition 2.6 A biclosure space is a triple (X, c_1, c_2) , where X is a set and c_1, c_2 are two closure operators on X .

Definition 2.7 A subset A of a biclosure space (X, c_1, c_2) is called closed if $c_1 c_2 A = A$. The complement of closed set is called open.

Proposition 2.8 A subset A of a biclosure space (X, c_1, c_2) is closed if and only if A is both closed subset of (X, c_1) and (X, c_2) . Furthermore, if A is a closed subset of a biclosure space (X, c_1, c_2) , then $c_1 c_2 A = A$ if and only if $c_1 A = A, c_2 A = A$.

Definition 2.9 Let (X, c_1, c_2) be a biclosure space. A biclosure space (Y, e_1, e_2) is called subspace of (X, c_1, c_2) if $Y \subseteq X$ and $e_i A = c_i A \cap Y$, for each $i \in \{1, 2\}$ and each $A \subseteq Y$.

Proposition 2.10 Let (X, c_1, c_2) be a biclosure space and let (Y, e_1, e_2) be a closed subspace of (X, c_1, c_2) . If F is a closed subset of (Y, e_1, e_2) , then F is a closed subset of (X, c_1, c_2) .

Lemma 2.11 Let (X, c_1, c_2) be a biclosure space and let (Y, e_1, e_2) be a closed subspace of (X, c_1, c_2) . If G is an open subset of (X, c_1, c_2) , then $G \cap Y$ is open in (Y, e_1) and an it is open in (Y, e_2) also.

Proposition 1.12 Let $\{(X_\lambda, c_\lambda^1, c_\lambda^2) : \lambda \in \Lambda\}$ be a family of biclosure spaces and let $\alpha \in \Lambda$. Then G is an open subset of $\{(X_\alpha, c_\alpha^1, c_\alpha^2)\}$ if and only if $G \times \prod_{\lambda \in (\Lambda - \{\alpha\})} X_\lambda$ is an open subset of $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^1, c_\lambda^2)$.

Definition 2.13 Let (X, c_1, c_2) and (Y, e_1, e_2) be two biclosure spaces and $i \in \{1, 2\}$. A function $f: (X, c_1, c_2) \rightarrow (Y, e_1, e_2)$ is said to be i -continuous if $f: (X, c_i) \rightarrow (Y, e_i)$ is continuous, and f is called continuous if $f: (X, c_i) \rightarrow (Y, e_i)$ is i -continuous for each $i \in \{1, 2\}$.

Definition 2.14 Let (X, c_1, c_2) and

(Y, e_1, e_2) be two biclosure spaces and $i \in \{1, 2\}$. A function $f: (X, c_1, c_2) \rightarrow (Y, e_1, e_2)$ is said to be i -closed (i -open) if $f: (X, c_i) \rightarrow (Y, e_i)$ is closed (open), and f is called closed (open) if $f: (X, c_i) \rightarrow (Y, e_i)$ is i -closed (i -open) for each $i \in \{1, 2\}$.

Definition 2.15 A biclosure space (X, c_1, c_2) is said to be a Hausdorff biclosure space if, whenever x and y are distinct points of X there exists an open subset U of (X, c_1) and an open subset V of (X, c_2) such that $x \in U, y \in V$ and $U \cap V = \phi$.

Definition 2.16 A biclosure space (X, c_1, c_2) is said to be a regular biclosure space if, for any closed subset F of (X, c_1) and any point $x \in X - F$, there exist disjoint open subsets U and V of (X, c_2) such that $x \in U$ and $F \subseteq V$.

Definition 2.17 ([10]) A biclosure space (X, c_1, c_2) is called a completely regular (briefly, C.R.) biclosure space if, for each point $x \in X$ and each closed subset F of (X, c_1) such that $x \notin F$, there exists a continuous function $f: (X, c_1, c_2) \rightarrow (I, c_1, c_2)$ such that $f(x) = 0$ and $f(F) = \{1\}$.

I. Urysohn Biclosure Space:

In this section, we introduce the concept of Urysohn biclosure spaces and we study some of their properties.

Definition 3.1 A biclosure space (X, c_1, c_2) is said to be a Urysohn biclosure space if, whenever x and y are distinct points of X there exists an open subset U of (X, c_1) and an open subset V of (X, c_2) such that $x \in U, y \in V$ and $c_1 U \cap c_2 V = \phi$.

Example 3.2 Let $X = \{a, b, c\}$ and define closure operators c_1 and c_2 on X by $c_i A = A$ for each $A \subseteq X$. Then (X, c_1, c_2) is a Urysohn biclosure space.

Proposition 3.3 Every Urysohn biclosure space is Hausdorff biclosure space.

Proof. Obvious.

Example 3.4 Let $c_1 = c_2$ be the closure

operator on Z^+ associated with the prime integer topology τ ([11, Example 61, p. 82]). Since the space (Z^+, τ) is a Hausdorff space, but not Urysohn space. Therefore, the biclosure space (Z^+, c_1, c_2) is Hausdorff, but not Urysohn.

We recall the following definitions from [8]

Definition 3.5 A subset A of a closure space (X, c) is said to be generalized closed (briefly, g-closed) if $cA \subseteq G$, whenever G is an open subset of (X, c) with $A \subseteq G$. A subset A of X is said to be a generalized open (briefly, g-open) set, if its complement is g-closed.

Definition 3.6 A biclosure space (X, c_1, c_2) is said to be generalized Hausdorff (briefly, g-Hausdorff) if for each two distinct points x and y of X , there exists a g-open subset U of (X, c_1) and g-open subset V of (X, c_2) such that $x \in U, y \in V$ and $U \cap V = \phi$.

Corollary 3.7 Every Urysohn biclosure space is a g-Hausdorff biclosure space.

Proof. It followed from Proposition 3.4 and ([8, Remark 1]).

The converse of the above corollary is not true in general, as the following example shows:

Example 3.8 Let X be a set with at least two points and let c_1 and c_2 be closure operators on X given by $c_1A = X = c_2A$ for all non-empty subset A of X and $c_1\phi = \phi = c_2\phi$. It easy to see that, the biclosure space (X, c_1, c_2) is g-Hausdorff, but not Urysohn.

It is easy to see that the biclosure space (X, c_1, c_2) of Example 3.8, is completely regular and regular, this with following example show that the concepts of Urysohnness and regularity of biclosure spaces are independent concepts.

Example 3.9 Consider the countable comple-ment extension topology \mathfrak{S} on R ([11, Example 63, p. 85]), and let $c_1 = c_2$ be the closure operator on R associated with the topology \mathfrak{S} on R . Since the space (R, \mathfrak{S}) is Uryshon, but not regular. So, the biclosure

space (R, c_1, c_2) is Urysohn, but not regular and hence it is not completely regular.

Theorem 3.10 If (Y, e_1, e_2) is a closed subspace of a Urysohn biclosure space (X, c_1, c_2) , then (Y, e_1, e_2) is a Urysohn biclosure space.

Proof. Let (Y, e_1, e_2) be a closed subspace of a Urysohn biclosure space (X, c_1, c_2) and let x and y be two distinct points of a biclosure space (Y, e_1, e_2) . Then x and y are two distinct points of a biclosure space (X, c_1, c_2) . Since (X, c_1, c_2) is a Urysohn biclosure space, there exists an open subset U of (X, c_1) and an open subset V of (X, c_2) such that $x \in U, y \in V$ and $c_1U \cap c_2V = \phi$, then, by Lemma 2.11, $U \cap Y$ is an open subset of (Y, e_1) and $V \cap Y$ is an open subset of (Y, e_2) containing x and y , respectively. Furthermore $e_1(U \cap Y) \cap e_2(V \cap Y) = c_1(U \cap Y) \cap c_2(V \cap Y) \cap Y \subseteq (c_1U \cap c_2V) \cap Y = \phi$. This means that (Y, e_1, e_2) is a Urysohn biclosure space.

Theorem 3.11 Let $\{(X_\lambda, c_\lambda^1, c_\lambda^2) : \lambda \in \Lambda\}$ be a family of biclosure spaces. Then $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^1, c_\lambda^2)$ is a Urysohn biclosure space if and only if $(X_\lambda, c_\lambda^1, c_\lambda^2)$ is a Urysohn biclosure space, for each $\lambda \in \Lambda$.

Proof. Suppose that $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^1, c_\lambda^2)$ is a Urysohn biclosure product space, and we assume $\alpha \in \Lambda$. To show $(X_\alpha, c_\alpha^1, c_\alpha^2)$ is Urysohn. Let x_α and y_α be two distinct points of X_α . Then $(x_\lambda)_{\lambda \in \Lambda}$ and $(y_\lambda)_{\lambda \in \Lambda}$ are two distinct points of $\prod_{\lambda \in \Lambda} X_\lambda$, where $x_\lambda = y_\lambda$ for each $\lambda \in \Lambda - \{\alpha\}$. By our hypothesis, there exist an open subset G of $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^1)$ and an open subset M of $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^{21})$ such that $(x_\lambda)_{\lambda \in \Lambda} \in G, (y_\lambda)_{\lambda \in \Lambda} \in M$ and $G \cap M = \phi$, but since $x_\lambda = y_\lambda$ for each $\lambda \in \Lambda - \{\alpha\}$, then any set contains x_λ it contains also y_λ for each $\lambda \in \Lambda - \{\alpha\}$. Therefore, it must exist an open subset U of (X_α, c_α^1) and an open subset V of (X_α, c_α^2) such that $x_\alpha \in U, y_\alpha \in V$ and $c_\alpha^1U \cap c_\alpha^2V = \phi$. Thus $(X_\alpha, c_\alpha^1, c_\alpha^2)$ is a Urysohn biclosure space.

Conversely; suppose that $(X_\lambda, c_\lambda^1, c_\lambda^2)$ is a

Urysohn biclosure space for each $\lambda \in \Lambda$. Let $(x_\lambda)_{\lambda \in \Lambda}$ and $(y_\lambda)_{\lambda \in \Lambda}$ be two distinct points of $\prod_{\lambda \in \Lambda} X_\lambda$. Then, there exists $\alpha \in \lambda$ such that $x_\alpha \neq y_\alpha$. Since $(X_\alpha, c_\alpha^1, c_\alpha^2)$ is a Urysohn biclosure space, there exist disjoint open subsets U of (X_α, c_α^1) and (X_α, c_α^2) such that $x_\alpha \in U, y_\alpha \in V$ and $c_\alpha^1 U \cap c_\alpha^2 V = \phi$. By Proposition 2.12 $U \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda$ is open in $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^1, c_\lambda^2)$ and $V \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda$ is open in $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^1, c_\lambda^2)$, such that $(x_\lambda)_{\lambda \in \Lambda} \in U \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda$, $(y_\lambda)_{\lambda \in \Lambda} \in V \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda$ and $\prod_{\lambda \in \Lambda} c_\lambda^1 \pi_\lambda (U \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda) \cap \prod_{\lambda \in \Lambda} c_\lambda^2 \pi_\lambda (V \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda) = (c_\alpha^1 U \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda) \cap (c_\alpha^2 V \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda) = (c_\alpha^1 U \cap c_\alpha^2 V) \times \prod_{\lambda \in \Lambda - \{\alpha\}} X_\lambda = \phi$. This means that $\prod_{\lambda \in \Lambda} (X_\lambda, c_\lambda^1, c_\lambda^2)$ is a Urysohn biclosure product space.

Proposition 3.12 Let $f: (X, c_1, c_2) \rightarrow (Y, e_1, e_2)$ be an injective continuous function. If (Y, e_1, e_2) is a Urysohn biclosure space, then (X, c_1, c_2) is Urysohn also.

Proof. Let a and b be two distinct points of X . Then $f(a)$ and $f(b)$ are distinct points of Y . Since (Y, e_1, e_2) is a Urysohn biclosure space, there exist disjoint open subsets U of (Y, e_1) and V of (Y, e_2) containing $f(a)$ and

$f(b)$ respectively such that $e_1 U \cap e_2 V = \phi$. Since f is continuous, so by Proposition 2.4, $f^{-1}(U)$ is an open subset of (X, c_1) containing a and $f^{-1}(V)$ is an open subset of (X, c_2) containing b . Again, by Propostion 2.4 and the continuity of f , we have $c_1 f^{-1}(U) \subseteq f^{-1}(e_1 U)$ and $c_2 f^{-1}(V) \subseteq f^{-1}(e_2 V)$. Therefore, $c_1 f^{-1}(U) \cap c_2 f^{-1}(V) = \phi$. Thus (X, c_1, c_2) is a urysohn biclosure space.

Theorem 3.13 Let $f: (X, c_1, c_2) \rightarrow (Y, e_1, e_2)$ be a continuous and open function. If (X, c_1, c_2) is a Urysohn biclosure space, then (Y, e_1, e_2) also.

Proof. Let $y_1 \neq y_2$ in Y . Then there exist $x_1 \neq x_2$ in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since (X, c_1, c_2) is a Urysohn space, there exists an open set U in (X, c_1) and open set V in (X, c_2) such that $x_1 \in U, x_2 \in V$ and $c_1 U \cap c_2 V = \phi$. Since f is an open function, $f(U)$ is open in (Y, e_1) containing y_1 and $f(V)$ is open in (Y, e_2) containing y_2 . Since f is continuous, $e_1 f(U) \cap e_2 f(V) \subseteq f(c_1 U \cap c_2 V) = \phi$. Hence (Y, e_1, e_2) is a Urysohn biclosure space.

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